Total Operators and Inhomogeneous Proper-Value Equations

José G. Vargas
PST Associates, LLC (USA)

138 Promontory Rd, Columbia, SC 29209
josegvargas@earthlink.net

To Professors Eckhard Hitzer, Zbigniew Oziewicz and Wolfgang Sproessig

Abstract. Kähler's two-sided angular momentum operator, $K+1$, is neither vector-valued nor bivector-valued. It is total in the sense that it involves terms for all three dimensions. Constant idempotents that are "proper functions" of $K+1$'s components are not proper functions of $K+1$. They rather satisfy "inhomogeneous proper-value equations", i.e. of the form $(K+1)U = \mu U + \pi$, where π is a scalar.

We consider an equation of this type with $K+1$ replaced with operators T that comprise $K+1$ as a factor, but also containing factors for both space and spacetime translations. We study the action of those T's on linear combinations of constant idempotents, so that only the algebraic (spin) part of $K+1$ has to be considered. π is now, in general, a non-scalar member of a Kähler algebra. We develop the system of equations to be satisfied by the combinations of those idempotents for which π becomes a scalar. We solve for its solutions with $\mu = 0$, which actually also makes $\pi = 0$.

The solutions with $\mu = \pi = 0$ all have three constituent parts, 36 of them being different in the ensemble of all such solutions. That set of different constituents is structured in such a way that we might as well be speaking of an algebraic representation of quarks. In this paper, however, we refrain from pursuing this identification in order to emphasize the purely mathematical nature of the argument.

Key words: Kähler algebra, Idempotents, Total operators, Inhomogeneous proper-value equations, quarks.

1 Introduction

This paper is a continuation of a previous one dealing with solutions of exterior systems [1], where we brought attention to Kähler's concept if total angular momentum operator, $K+1$. We then introduced the tensor product of tangent and cotangent Clifford algebras of spacetime (cotangent here

refers to differential forms viewed as functions of r-surfaces, not as antisymmetric multilinear functions of vectors). In this new arena, we resorted to the action of the operator $K+1$, which thus makes our study purely algebraic. We reached idempotents of the type $\boldsymbol{\epsilon}^{\pm}\boldsymbol{I}_{ij}^{\pm}\boldsymbol{P}_{l}^{\pm}$, where ij is $(1,2)$ or $(2,3)$ or $(3,1)$ without correlations among the superscripts. We refer readers to [1] for the definition of $\epsilon^{\pm}$, $\boldsymbol{I}_{ij}^{\pm}$ and $\boldsymbol{P}_{l}^{\pm}$.

There are 72 such expressions when all combinations of indices are taken into account. Symmetries reduce their number to 48. A further reduction of those 48 to 36 follows through the study of proper values, in an extended sense of the term, when the operator is a total operator for rotations and spacetime translations. The steps of this study are as follows.

In section 2, we continue the discussion of the nature of total angular momentum in Kähler's calculus of differential forms. In section 3, we consider the emergence of algebraic inhomogeneous proper value operation through the action of $K+1$ on idempotents. In section 4, we go one step further in the study on the same idempotents by acting on them with the product of $K+1$ with the operators /or translations along the axes.

In section 5, we consider the action of a second total operators, $(K+1)d\boldsymbol{r}$, where $d\boldsymbol{r}$ is the translation element in 3-D Euclidean space. It is total for the displacements like $K+1$ is total for the rotations. In section 6, we directly address the problem of finding linear combinations of the aforementioned parts, combinations for which we have a proper value and co-value. In the middle of the argument, we specialize to proper value zero.

In section 7, we make our operator still more total by including time translations. In section 8, we raise the issue of using these purely mathematical results to formulate a physics of particles with it.

2 Kähler's Total Angular Momentum

We have presented in English much of Kähler's treatment (in German) of components of angular momentum, and then of total angular momentum, respectively in [2] and [1]. The argument starts with his approach to Lie differentiation as a sophistical case of partial differentiation.

Let U_r be the differential form

$$U_r := a_{i_1 \dots i_r} dx^{i_1 \dots i_r}, \tag{1}$$

where the coordinates are Cartesian. Consider $\partial U/\partial \phi$, where ϕ is the azimuthal coordinate. As explained by Kähler [3], the Lie derivative of U_r with respect to ϕ is simply the $\partial U_r/\partial \phi$. But $\partial U_r/\partial \phi$ is not as in the literature,

as we proceed to explain. We should not only differentiate the $a_{i_1\ldots i_r}$, but also the dx^i's, using that $\partial(dx^i)/\partial\phi = d(\partial x^i/\partial\phi)$.

Since $\partial/\partial\phi$ is of the form $f^l(x)\partial/\partial x^l$, the preceding consideration implies that the action of operators $f^l(x)/\partial/\partial x^l$ on differential forms (1) have terms additional to those where one simply differentiates the components, which is worth remembering since this is counterintuitive. We shall not be distracted here with details. Suffice to say that, if the $f^l(x)$ are constants, then

$$\left[f^l\frac{\partial}{\partial x^l}\right]U = f^l\frac{\partial a_{i_i\ldots i_r}}{\partial x^l}dx^{i_1\ldots i_r}. \tag{2}$$

When they are not constant, we shall distinguish between $\left[g^l\frac{\partial}{\partial x^m}\right]U$ and $g^l\left[\frac{\partial U}{\partial x^m}\right]$, as they are not equal. The $\partial U/\partial x^m$ is in itself given by (2) with $f^l = 0$, except for one of them, $f^m = 1$.

Another important remark is that, in the Kähler calculus, U_r is a function of hypersurfaces. It does not depend on the coordinates used for its description. And $\partial U_r/\partial\phi$ will be the total derivative $dU_r/d\phi$ if ϕ constitutes one of the coordinates, y_n, in a coordinate system (y) where all the other coordinates are the $n-1$ independent constants of the motion not additive to y^n in the system,

$$\frac{dx^i}{dy^n} = a^i(x^1,\ldots,x^n). \tag{3}$$

See [5].

A moderately long process starting with the foregoing concepts leads Kähler to obtain the so called components of angular momentum operator in 3-D dimensional space, E_3, as

$$\frac{\partial U}{\partial\phi^i} = \chi_i U := \left(x^j\frac{\partial U}{\partial x^k} - x^k\frac{\partial U}{\partial x^j}\right) + \frac{1}{2}w^i\vee U - \frac{1}{2}U\vee w^i, \tag{4}$$

where (i,j,k) are the three cyclic permutations (identity included) of $(1,2,3)$, where "$\vee$" stands for Clifford product, and where the $w_i(= w^i)$ are the dx^{jk} $(=: dx^j\wedge dx^k)$. See [3] and [4].

Except for the unit imaginary factor, the first half on the right of (4) coincides with the components of orbital angular momentum in quantum mechanics (with $\hbar = 1$).

In view of what we have said, the ϕ^i are azimuthal coordinates relative to different axes and, therefore, belonging to different coordinate systems.

Kähler defines a total angular momentum operator, $K+1$, as the two-sided operator whose action on U is given by

$$(K+1)U = J_l U w^l, \tag{5}$$

with summation over repeated indices. It is linear not in the sense of linearity in a vector space, but in the sense that it is a sum of terms each of which contains one and only one of the J_l, the w^l being a basis of differential 2-forms. Applying $(K+1)$ to (5) and performing operations, one gets

$$(K+1)^2 U = -\sum_l \chi_l^2 U + (K+1)U. \tag{6}$$

Kähler apparently chose the symbol $K+1$ so that one gets $K(K+1)$ when one takes to the left the last term in (6). The minus sign in front of the first term on the right has to do with the fact that χ_i does not contain the unit imaginary as factor.

We shall be applying the operator $K+1$ to Clifford valued differential form. It is a differential operator in the Kähler subalgebra of scalar-valued differential forms.

3 Inhomogeneous Proper Value Equations of $K+1$

In order to have notational continuity with [1], we shall use the symbol J_l instead of X_l. Now as then, $d\boldsymbol{x}^l$ will refer to $dx^l\mathbf{a}_l$ (no sum), with $l = i, j, k$ and $\mathbf{a}_l = \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{x}$. $d\boldsymbol{x}^l$ belongs to the commutative algebra that we defined in [1]; the J_l and $K+1$ do not. The action on constant differentials – as our idempotents will be – of the orbital part of angular momentum operators is zero. Thus the action of the J_l and $K+1$ is purely algebraic. This action is zero for scalar and for $d\boldsymbol{x}^{ijk}(\equiv dx^{ijk}\boldsymbol{a}_i\boldsymbol{a}_j\boldsymbol{a}_k)$, since dx^{ijk} commutes with the whole algebra of scalar-valued differential forms and it, therefore, commutes with J_l and $K+1$.

In [1], we obtained

$$J_l d\boldsymbol{x}^1 = (0, dx^3, -dx^2)\boldsymbol{a}_1. \tag{7}$$

We similarly have

$$J_l d\boldsymbol{x}^2 = (-d\boldsymbol{x}^3, 0, d\boldsymbol{x}^1), \qquad J_l d\boldsymbol{x}^3 = (d\boldsymbol{x}^2, -d\boldsymbol{x}x^1, 0). \tag{8}$$

Straightforward computations yields

$$J_i d\boldsymbol{x}^{jk} = 0, \tag{9}$$

and, without summation over repeated indices,

$$J_i d\boldsymbol{x}^{ki} = \frac{1}{2}(w^i d\boldsymbol{x}^{ki} - d\boldsymbol{x}^{ki} w^i) = -w^k \boldsymbol{a}_{ki} = w^k \boldsymbol{a}_{ik}, \tag{10}$$

and

$$J_i d\boldsymbol{x}^{ij} = \frac{1}{2}(w^i w^k - w^k w^i)\boldsymbol{a}_{ij} = w^j \boldsymbol{a}_{ij}. \tag{11}$$

In general, a repeated index i, j or k will not mean summation, For that, we use the index l, or m.

For later use, it is convenient to rewrite (9)-(11) by going over the three components of the operator, acting on the same differential form:

$$J_i d\boldsymbol{x}^{jk} = 0, \quad J_j d\boldsymbol{x}^{jk} = w^k \boldsymbol{a}_{jk}, \quad J_k d\boldsymbol{x}^{jk} = w^j \boldsymbol{a}_{kj}. \tag{12}$$

As in [1], we define

$$\boldsymbol{I}^{\pm}_{ij} = \frac{1}{2}(1 \pm d\boldsymbol{x}^{ij}). \tag{13}$$

Then

$$J_i \boldsymbol{I}^{\pm}_{jk} = 0, \quad J_i \boldsymbol{I}^{\pm}_{ki} = \pm\frac{1}{2} w^k \boldsymbol{a}_{ik}, \quad J_i \boldsymbol{I}^{\pm}_{ij} = \pm\frac{1}{2} w^j \boldsymbol{a}_{ij}, \tag{14}$$

obviously without summation over repeated indices. For fixed subscripts of $\boldsymbol{I}^{\pm}$ and different subscripts of J, we have

$$J_i \boldsymbol{I}_{jk} = 0, \quad J_j \boldsymbol{I}^{\pm}_{jk} = \pm\frac{1}{2} w^k \boldsymbol{x}_{jk}, \quad J_k \boldsymbol{I}^{\pm}_{jk} w^j \boldsymbol{a}_{jk}. \tag{15}$$

Since $w^j w^i = w^k$ and $w^i \boldsymbol{a}_{jk} = d\boldsymbol{x}^{jk}$, we have

$$J_j \boldsymbol{I}^{\pm}_{jk} = \pm\frac{1}{2} w^j w^i \boldsymbol{a}_{jk} = -\frac{wj}{2} + \frac{wj}{2} \pm \frac{1}{2} w^j d\boldsymbol{x}^{jk} = w^j (\boldsymbol{I}^{\pm}_{jk} - \frac{1}{2}), \tag{16}$$

and, in a similar way,

$$J_k \boldsymbol{I}^{\pm}_{jk} = w^k (\boldsymbol{I}^{\pm}_{jk} - \frac{1}{2}). \tag{17}$$

In order to compute $(K+1)\boldsymbol{I}^{\pm}_{ij}$, it is best to use (15) in (5):

$$(K+1)\boldsymbol{I}^{\pm}_{ij} = J_l \boldsymbol{I}^{\pm}_{ij} w^l = \pm w^k \boldsymbol{a}_{ij} = \pm d\boldsymbol{x}^{ij} = 2\boldsymbol{I}^{\pm}_{ij} - 1, \tag{18}$$

of which we say that it is an inhomogeneous proper values equation. We refer to -1 as its co-value. On the other hand

$$(K+1)d\boldsymbol{x}^{ij} = (K+1)(2\boldsymbol{I}^{+}_{ij} - 1) = 2(K+1)\boldsymbol{I}^{+}_{ij} = 2d\boldsymbol{x}^{ij}, \tag{19}$$

which is homogeneous.

We obtained in [1]:

$$(K+1)d\boldsymbol{x}^l = 2d\boldsymbol{x}^l, \tag{20}$$

and defined

$$\boldsymbol{P}_l^{\pm} = \frac{1}{2}(1 \pm d\boldsymbol{x}^l). \tag{21}$$

Clearly

$$(K+1)\boldsymbol{P}_l^{\pm} = (K+1)(\pm\frac{d\boldsymbol{x}^l}{2}) = \pm d\boldsymbol{x}^l = 2\boldsymbol{P}_l^{\pm} - 1, \tag{22}$$

where value and co-value are as in (18)-(19).

The $\boldsymbol{P}_l^{\pm}$ are proper functions of $d\boldsymbol{x}^l$ acting on the left. But $d\boldsymbol{x}^l$ is not a component of $d\boldsymbol{r}$; dx^l is. The role of $d\boldsymbol{x}^i\boldsymbol{a}_i$ (no sum) is played by $J_i w^i$:

$$d\boldsymbol{x}^i U \to J_i U w^i \qquad \text{(no sum)}.$$

4 First Results for Combined Rotations and Translations

We have seen that the $\boldsymbol{I}_{ij}^{\pm}$ are proper functions with proper value zero of J_k. But they are not proper functions of J_i, J_j and $K+1$. We now study the action of $K+1$ on idempotents $\boldsymbol{I}_{ij}\boldsymbol{P}_l$. It makes a great difference whether l is or is not equal to one of the subscripts of $\boldsymbol{I}_{ij}$. Key for fluid computations are equations (19) and (20), as well as $(K+1)1 = 0$ and $(K+1)d\boldsymbol{x}^{ijk} = 0$.

We easily have

$$(K+1)\boldsymbol{I}_{ij}^{+}\boldsymbol{P}_i^{\pm} = (K+1)\frac{1}{4}(1 + d\ldots \pm d\ldots \pm d\ldots) = \frac{1}{2}(d\ldots \pm d\ldots \pm d\ldots). \tag{23}$$

We have used suspension marks to indicate the irrelevance of the details provided that none of those suspension marks is for $d\boldsymbol{x}^{ijk}$. We thus have, adding and subtracting 1/2,

$$(K+1)\boldsymbol{I}_{ij}^{+}\boldsymbol{P}_i^{\pm} = \frac{1}{2}(1 + d\ldots \pm d\ldots \pm d\ldots) - \frac{1}{2} = 2\boldsymbol{I}_{ij}^{\pm}\boldsymbol{P}_i^{\pm} - \frac{1}{2}. \tag{24}$$

Similarly,

$$(K+1)\boldsymbol{I}_{ij}^{-}\boldsymbol{P}_i^{\pm} = 2\boldsymbol{I}_{ij}^{-}\boldsymbol{P}_i^{\pm} - \frac{1}{2}. \tag{25}$$

Consider next $(K+1)\boldsymbol{I}_{ij}^{+}\boldsymbol{P}_k^{\pm}$. We have

$$\begin{aligned}(K+1)\boldsymbol{I}_{ij}^{+}\boldsymbol{P}_k^{\pm} &= (K+1)\frac{1}{4}(1 + d\ldots \pm d\ldots \pm d\boldsymbol{x}^{ijk}) = \frac{1}{2}(d\ldots \pm d\ldots) = \\ &= \frac{1}{2}(1 + d\ldots \pm d\ldots \pm d\boldsymbol{x}^{ijk}) - \frac{1}{2}(1 \pm d\boldsymbol{x}^{ijk}) = 2\boldsymbol{I}_{ij}^{+}P_k^{\pm} - \frac{1}{2}(I \pm d\boldsymbol{x}^{ijk}).\end{aligned} \tag{26}$$

Proceeding similarly, we get

$$(K+1)\boldsymbol{I}^{-}_{ij}\boldsymbol{P}^{\pm}_{k} = 2\boldsymbol{I}^{-}_{ij}\boldsymbol{P}^{\pm}_{k} - \frac{1}{2}(1 \pm d\boldsymbol{x}_{ijk}). \tag{27}$$

We next want to know the joint effect of $K+1$ and one of the components of $d\boldsymbol{r}$ (in the sense of $d\boldsymbol{x}^l$, not dx^l). Since $d\boldsymbol{x}^i\boldsymbol{I}_{ij}\boldsymbol{P}_k$ does contain neither 1 nor $d\boldsymbol{x}^{ijk}$, it follows that

$$[(K+1)d\boldsymbol{x}^i]\boldsymbol{I}^{+}_{ij}\boldsymbol{P}^{\pm}_{k} = 2d\boldsymbol{x}^i\boldsymbol{I}^{+}_{ij}\boldsymbol{P}^{\pm}_{k}, \tag{28a}$$

$$[(K+1)d\boldsymbol{x}^i]\boldsymbol{I}^{-}_{ij}\boldsymbol{P}^{\pm}_{k} = 2d\boldsymbol{x}^i\boldsymbol{I}^{+}_{ij}\boldsymbol{P}^{\pm}_{k}. \tag{28b}$$

On the other hand, $d\boldsymbol{x}^k\boldsymbol{I}^{+}_{ij}\boldsymbol{P}^{\pm}_{i}$ has a $d\boldsymbol{x}^{ijk}$ term, but not a scalar term. Hence

$$[(K+1)d\boldsymbol{x}^k]\boldsymbol{I}^{+}_{ij}\boldsymbol{P}^{\pm}_{i} = 2d\boldsymbol{x}^k\boldsymbol{I}^{+}_{ij}\boldsymbol{P}^{\pm}_{k} - \frac{1}{2}d\boldsymbol{x}^{ijk}, \tag{29a}$$

$$[(K+1)d\boldsymbol{x}^k]\boldsymbol{I}^{-}_{ij}\boldsymbol{P}^{\pm}_{i} = 2d\boldsymbol{x}^k\boldsymbol{I}^{-}_{ij}\boldsymbol{P}^{\pm}_{k} - \frac{1}{2}d\boldsymbol{x}^{ijk}. \tag{29b}$$

We now let the subscript of the $d\boldsymbol{x}$ factor inside the bracket coincide with the subscript of $\boldsymbol{P}^{\pm}$, so that it is absorbed. One immediately sees whether scalar and $d\boldsymbol{x}^{ijk}$ terms are present. In the absorption process, a minus sign may appear. One readily obtains

$$[(K+1)d\boldsymbol{x}^i]\boldsymbol{I}^{+}_{ij}\boldsymbol{P}^{\pm}_{i} = \pm(2\boldsymbol{I}^{+}_{ij}\boldsymbol{P}^{\pm}_{i} - \frac{1}{2}), \tag{30a}$$

$$[(K+1)d\boldsymbol{x}^i]\boldsymbol{I}^{-}_{ij}\boldsymbol{P}^{\pm}_{i} = \pm(2\boldsymbol{I}^{-}_{ij}\boldsymbol{P}^{\pm}_{i} - \frac{1}{2}), \tag{30b}$$

$$[(K+1)d\boldsymbol{x}^k]\boldsymbol{I}^{+}_{ij}\boldsymbol{P}^{\pm}_{k} = \pm[2\boldsymbol{I}^{+}_{ij}\boldsymbol{P}^{\pm}_{k} \mp \frac{1}{2}(1 + d\boldsymbol{x}^{ijk})], \tag{31a}$$

$$[(K+1)d\boldsymbol{x}^k]\boldsymbol{I}^{-}_{ij}\boldsymbol{P}^{\pm}_{k} = \pm[2\boldsymbol{I}^{-}_{ij}\boldsymbol{P}^{\pm}_{k} \mp \frac{1}{2}(1 + d\boldsymbol{x}^{ijk})]. \tag{31b}$$

Equations (30a) and (30b) are inhomogeneous proper value equations. All the others are not. Equations (28) are proper value equations of $(K+1)$, not of $(K+1)d\boldsymbol{x}^i$.

We have not yet integrated $K+1$ with total translation operator.

5 Total Operator for 3-D Euclidean Space

We shall now deal with the action of operator $(K+1)d\boldsymbol{r}$ on the idempotents. In principle (it would be a tedious check), they do not commute. The question then could be: Why $(K+1)d\boldsymbol{r}$ instead of $d\boldsymbol{r}(K+1)$. We leave the study of the alternative to interested readers.

For any two different indices i and j, we have

$$\boldsymbol{I}_{ij}^{+}\boldsymbol{P}_{i}^{\pm} = \boldsymbol{I}_{ij}^{+}\boldsymbol{P}_{j}^{\pm}, \;\; \boldsymbol{O}\boldsymbol{I}_{ij}^{-}\boldsymbol{P}_{i}^{+} = \boldsymbol{I}_{ij}^{-}\boldsymbol{P}_{j}^{\mp}. \tag{31}$$

This leads us to consider the pairs

$$(\boldsymbol{I}_{12}^{+}\boldsymbol{P}_{1}^{+}, \boldsymbol{I}_{12}^{+}\boldsymbol{P}_{2}^{+}), (\boldsymbol{I}_{12}^{+}\boldsymbol{P}_{1}^{-}, \boldsymbol{I}_{12}^{+}\boldsymbol{P}_{2}^{-}), (\boldsymbol{I}_{12}^{-}\boldsymbol{P}_{1}^{+}, \boldsymbol{I}_{12}^{-}\boldsymbol{P}_{2}^{-}), (\boldsymbol{I}_{12}^{-}\boldsymbol{P}_{1}^{-}, \boldsymbol{I}_{12}^{-}\boldsymbol{P}_{2}^{+}), \tag{32}$$

of equal idempotents. We nevertheless do not discard half of them (which ones in the first place?) since they are associated with different directions for space translations. This implies that they would be associated with different exponential factors in solutions with symmetry of exterior systems [1]. Both copies will be used when we shall later invoke time translations.

When doing purely algebraic computations with them, we may use, however, just a set of four different ones. It does not matter which set we choose. We shall consider the four $\boldsymbol{I}_{12}^{\pm}\boldsymbol{P}_{1}^{*}$ together with $\boldsymbol{I}_{12}^{\pm}\boldsymbol{P}_{3}^{*}$. We denote them as X_A, $A = 1, \ldots, 8$, the ordering being as in table 1.

The total translation operator is $d\boldsymbol{r}$, which we can write as $d\boldsymbol{r}' + d\boldsymbol{x}^3$, where $d\boldsymbol{r}' = d\boldsymbol{x}^1 + d\boldsymbol{x}^2$. Notice that

$$d\boldsymbol{r}'\boldsymbol{I}_{12}^{-} = (d\boldsymbol{x} + d\boldsymbol{y})\boldsymbol{I}_{12}^{-} = 2d\boldsymbol{x}\boldsymbol{I}_{12}^{+}\boldsymbol{I}_{12}^{-} = 0 \tag{33}$$

and also that

$$d\boldsymbol{r}'\boldsymbol{I}_{12}^{+} = d\boldsymbol{x}^1\boldsymbol{I}_{12}^{+} + d\boldsymbol{x}^2\boldsymbol{I}_{12}^{+} = 2d\boldsymbol{x}^1\boldsymbol{I}_{12}^{+}. \tag{34}$$

Hence,

$$d\boldsymbol{r}'\boldsymbol{I}_{12}^{+}\boldsymbol{P}_{1}^{\pm} = 2d\boldsymbol{x}^1\boldsymbol{I}_{12}^{+}\boldsymbol{P}_{1}^{\pm} = \pm 2\boldsymbol{I}_{12}^{+}\boldsymbol{P}_{1}^{\pm}. \tag{35}$$

On the basis of those considerations we build table 1.

Table 1. Action of $d\boldsymbol{r}$ on the $\boldsymbol{I}_{12}^{\pm}\boldsymbol{P}_1^*$ and $\boldsymbol{I}_{12}^{\pm}\boldsymbol{P}_3^*$.

X_A	$d\boldsymbol{r}^1 X_A + d\boldsymbol{x}^3 X_A$
$\boldsymbol{I}_{12}^{+}\boldsymbol{P}_1^{+} = \frac{1}{4}(1 + d\boldsymbol{x}^1 + d\boldsymbol{x}^2 + d\boldsymbol{x}^{12})$	$\frac{1}{2}(1 + d\boldsymbol{x}^1 + d\boldsymbol{x}^2 + d\boldsymbol{x}^{12})$ $+\frac{1}{4}(d\boldsymbol{x}^3 + d\boldsymbol{x}^{13} + d\boldsymbol{x}^{23} + d\boldsymbol{x}^{123})$
$\boldsymbol{I}_{12}^{+}\boldsymbol{P}_1^{-} = \frac{1}{4}(1 - d\boldsymbol{x}^1 - d\boldsymbol{x}^2 + d\boldsymbol{x}^{12})$	$\frac{1}{2}(-1 + d\boldsymbol{x}^1 + d\boldsymbol{x}^2 - d\boldsymbol{x}^{12})$ $+\frac{1}{4}(d\boldsymbol{x}^3 - d\boldsymbol{x}^{13} - d\boldsymbol{x}^{23} + d\boldsymbol{x}^{123})$
$\boldsymbol{I}_{12}^{-}\boldsymbol{P}_1^{+} = \frac{1}{4}(1 + d\boldsymbol{x}^1 - d\boldsymbol{x}^2 - d\boldsymbol{x}^{12})$	$O + \frac{1}{4}(d\boldsymbol{x}^3 + d\boldsymbol{x}^{13} - d\boldsymbol{x}^{23} - d\boldsymbol{x}^{123})$
$\boldsymbol{I}_{12}^{-}\boldsymbol{P}_1^{-} = \frac{1}{4}(1 - d\boldsymbol{x}^1 + d\boldsymbol{x}^2 - d\boldsymbol{x}^{12})$	$O + \frac{1}{4}(d\boldsymbol{x}^3 - d\boldsymbol{x}^{13} + d\boldsymbol{x}^{23} - d\boldsymbol{x}^{123})$
$\boldsymbol{I}_{12}^{+}\boldsymbol{P}_3^{+} = \frac{1}{4}(1 + d\boldsymbol{x}^3 + d\boldsymbol{x}^{12} + d\boldsymbol{x}^{123})$	$\frac{1}{2}(d\boldsymbol{x}^1 + d\boldsymbol{x}^{13} + d\boldsymbol{x}^2 + d\boldsymbol{x}^{23})$ $+\frac{1}{4}(1 + d\boldsymbol{x}^3 + d\boldsymbol{x}^{12} + d\boldsymbol{x}^{123})$
$\boldsymbol{I}_{12}^{+}\boldsymbol{P}_3^{-} = \frac{1}{4}(1 - d\boldsymbol{x}^3 + d\boldsymbol{x}^{12} - d\boldsymbol{x}^{123})$	$\frac{1}{2}(d\boldsymbol{x}^1 - d\boldsymbol{x}^{13} + d\boldsymbol{x}^2 - d\boldsymbol{x}^{23})$ $+\frac{1}{4}(-1 + d\boldsymbol{x}^3 - d\boldsymbol{x}^{12} + d\boldsymbol{x}^{123})$
$\boldsymbol{I}_{12}^{-}\boldsymbol{P}_3^{+} = \frac{1}{4}(1 + d\boldsymbol{x}^3 - d\boldsymbol{x}^{12} - d\boldsymbol{x}^{123})$	$O + \frac{1}{4}(1 + d\boldsymbol{x}^3 - d\boldsymbol{x}^{12} - d\boldsymbol{x}^{123})$
$\boldsymbol{I}_{12}^{-}\boldsymbol{P}_3^{-} = \frac{1}{4}(1 - d\boldsymbol{x}^3 - d\boldsymbol{x}^{12} + d\boldsymbol{x}^{123})$	$O + \frac{1}{4}(-1 + d\boldsymbol{x}^3 + d\boldsymbol{x}^{12} - d\boldsymbol{x}^{123})$

We use this table in the next section.

Consider the equation

$$[(K+1)d\boldsymbol{r}]X_A = \mu' X_A + \pi_A. \tag{36}$$

Needless to say that μ' and π_A are not fully determined and that, to the extent that they are, π_A will not in general be a number. We shall make linear combinations that are. Start by defining

$$\mu := -\frac{\mu'}{4} \tag{37}$$

so that Eq. (37) can be given the form

$$[(K+1)d\boldsymbol{r} + 4\mu]X_A = \pi_A. \tag{38}$$

The factor $-1/4$ in (38) has been chosen to facilitate computations.

We form linear combinations $\Sigma_A \lambda_A X_A$. For the action of $(K+1)d\boldsymbol{r}$ on these combinations, we make $K+1$ act on the second column of table 1. the latter's action on 1 and $d\boldsymbol{r}^{123}$ is zero, and the action on the other elements will simply multiply them by two. We then proceed to compute

$$\lambda_A[(K+1)d\boldsymbol{r} + 4\mu]X_A.$$

and sum all that up. We arrange the resulting coefficients of the $\boldsymbol{dx}^l$, $\boldsymbol{dr}^{lh}$ and $\boldsymbol{dr}^{123}$ in columns for easy summation. We leave the scalars for last.

Table 2. $\lambda_A[(K+1)\boldsymbol{dr}+4\mu]X_A$

$\boldsymbol{dx}^1$	$\boldsymbol{dx}^2$	$\boldsymbol{dx}^3$	$\boldsymbol{dx}^{12}$	$\boldsymbol{dx}^{13}$	$\boldsymbol{dx}^{23}$	$\boldsymbol{dx}^{123}$
$\lambda_1+\lambda_1\mu$	$\lambda_1+\lambda_1\mu$	$\frac{\lambda_1}{2}$	$\lambda_1+\lambda_1\mu$	$\frac{\lambda_1}{2}$	$\frac{\lambda_1}{2}$	$\frac{\lambda_1}{2}$
$\lambda_2-\lambda_2\mu$	$\lambda_2-\lambda_2\mu$	$\frac{\lambda_2}{2}$	$-\lambda_2+\lambda_2\mu$	$-\frac{\lambda_2}{2}$	$-\frac{\lambda_2}{2}$	$\frac{\lambda_2}{2}$
$\lambda_3\mu$	$-\lambda_3\mu$	$\frac{\lambda_3}{2}$	$-\lambda_3\mu$	$\frac{\lambda_3}{2}$	$-\frac{\lambda_3}{2}$	$-\frac{\lambda_3}{2}$
$-\lambda_4\mu$	$\lambda_4\mu$	$\frac{\lambda_4}{2}$	$-\lambda_4\mu$	$-\frac{\lambda_4}{2}$	$\frac{\lambda_4}{2}$	$-\frac{\lambda_4}{2}$
λ_5	λ_5	$\frac{\lambda_5}{2}+\lambda_5\mu$	$\frac{\lambda_5}{2}+\lambda_5\mu$	λ_5	λ_5	$\frac{\lambda_5}{2}+\lambda_5\mu$
λ_6	λ_6	$\frac{\lambda_6}{2}-\lambda_6\mu$	$-\frac{\lambda_6}{2}+\lambda_6\mu$	$-\lambda_6$	$-\lambda_6$	$\frac{\lambda_6}{2}-\lambda_2\mu$
0	0	$\frac{\lambda_7}{2}+\lambda_7\mu$	$-\frac{\lambda_7}{2}-\lambda_7\mu$	0	0	$-\frac{\lambda_7}{2}-\lambda_7\mu$
0	0	$\frac{\lambda_8}{2}-\lambda_8\mu$	$\frac{\lambda_8}{2}-\lambda_8\mu$	0	0	$-\frac{\lambda_8}{2}+\lambda_8\mu$

6 Search for Solutions of Inhomogeneous Proper Values Equations of $(K+1)\boldsymbol{dr}$

Solutions of inhomogeneous proper value equations of the operator $(K+1)\boldsymbol{dr}$ are obtained by adding each column in table 2 and setting the sums to zero. In what way, we shall have that a surviving scalar of the left of

$$\sum_A \lambda_A[(K+1)\boldsymbol{dr}+4\mu]X_A = \sum_A \pi_A, \tag{39}$$

will equal the right hand side. Hence $\sum \pi_A$ will be the co-value of the equation

$$[(K+1)\boldsymbol{dr}][\sum_1^8 \lambda_A X_A] = \mu' \sum \lambda_A \chi_A + \pi, \tag{40}$$

where $\pi (:= \sum \pi_A)$ is a number.

Equating to zero the coefficients of $\boldsymbol{dx}^1$ and $\boldsymbol{dx}^2$, we respectively have

$$(\lambda_1+\lambda_2)+(\lambda_5+\lambda_6)+\mu[(\lambda_1-\lambda_2)+(\lambda_3-\lambda_4)]=0, \tag{41}$$

$$(\lambda_1+\lambda_2)+(\lambda_5+\lambda_6)+\mu[(\lambda_1-\lambda_2)-(\lambda_3-\lambda_4)]=0. \tag{42}$$

The pair of equations (21)-(22) is equivalent to the pair

$$\lambda_4=\lambda_3 \tag{43}$$

$$(\lambda_1+\lambda_2)+(\lambda_5+\lambda_6)+\mu(\lambda_1-\lambda_2)=0. \tag{44}$$

Consider next the pair of equations for $d\boldsymbol{x}^{13}$ and $d\boldsymbol{x}^{23}$:

$$\frac{1}{2}[(\lambda_1 - \lambda_2) + (\lambda_3 - \lambda_4)] + (\lambda_5 - \lambda_6) = 0, \tag{45}$$

$$\frac{1}{2}[(\lambda_1 - \lambda_2) - (\lambda_3 - \lambda_4)] + (\lambda_5 - \lambda_6) = 0. \tag{46}$$

In view of (43), they both become

$$(\lambda_1 - \lambda_2) + 2(\lambda_5 - \lambda_6) = 0. \tag{47}$$

From the terms in $d\boldsymbol{x}^{123}$ and $d\boldsymbol{x}^{3}$, we respectively get

$$(\lambda_1+\lambda_2)-(\lambda_3+\lambda_4)+(\lambda_5+\lambda_6)-(\lambda_7+\lambda_8)+2\mu[(\lambda_5-\lambda_6)-(\lambda_7-\lambda_8)] = 0, \tag{48}$$

$$(\lambda_1+\lambda_2)+(\lambda_3+\lambda_4)+(\lambda_5+\lambda_6)+(\lambda_7+\lambda_8)+2\mu[(\lambda_5-\lambda_6)+(\lambda_7-\lambda_8)] = 0. \tag{49}$$

Adding and subtracting (48) and (49), we get

$$(\lambda_1 + \lambda_2) + (\lambda_5 + \lambda_6) + 2\mu(\lambda_5 - \lambda_6) = 0 \tag{50}$$

$$(\lambda_3 + \lambda_4) + (\lambda_7 + \lambda_8) + 2\mu(\lambda_7 - \lambda_8) = 0. \tag{51}$$

Finally, from the equation for $d\boldsymbol{x}^{12}$, we get

$$\begin{aligned}(\lambda_1 - \lambda_2) + \frac{1}{2}(\lambda_5 - \lambda_6) - \frac{1}{2}(\lambda_7 - \lambda_8) + \\ + \mu[(\lambda_1 + \lambda_2) - (\lambda_3 + \lambda_4) + (\lambda_5 + \lambda_6) - (\lambda_7 + \lambda_8)] = 0.\end{aligned} \tag{52}$$

We proceed to solve this system of equations. From (44) and (50):

$$\mu(\lambda_1 - \lambda_2) - 2\mu(\lambda_5 - \lambda_6) = 0. \tag{53}$$

Hence, either $\mu = 0$ or

$$\lambda_1 - \lambda_2 = 2(\lambda_5 - \lambda_6). \tag{54}$$

We develop the option $\mu = 0$ and leave the option $\mu \neq 0$ for a future paper

Equations (47) and (54) become

$$\lambda_5 + \lambda_6 = -\lambda_1 - \lambda_2, \tag{55}$$

$$\lambda_5 - \lambda_6 = -\frac{\lambda_1}{2} + \frac{\lambda_2}{2}. \tag{56}$$

Hence

$$\lambda_5 = -\frac{3}{4}\lambda_1 - \frac{1}{4}\lambda_2, \qquad \lambda_6 = -\frac{1}{4}\lambda_1 - \frac{3}{4}\lambda_2. \tag{57}$$

From (51) and (43), we obtain

$$\lambda_7 + \lambda_8 = -2\lambda_3, \tag{58}$$

and from (52) and (56), we further obtain

$$\lambda_7 - \lambda_8 = \frac{3}{2}\lambda_1 - \frac{3}{2}\lambda_2. \tag{59}$$

Adding and substracting, we get

$$\lambda_7 = \frac{3}{4}\lambda_1 - \frac{3}{4}\lambda_2 - \lambda_3, \tag{60}$$

$$\lambda_8 = -\frac{3}{4}\lambda_1 + \frac{3}{4}\lambda_2 - \lambda_3. \tag{61}$$

This solution contains two simpler, complementary ones, The first one is given by $\lambda_2 = \lambda_1 = 1$, $\lambda_3 = \lambda_4 = 0$. Then $\lambda_5 = \lambda_6 = -1$, $\lambda_7 = \lambda_8 = 0$. We see three parts in it

$$\boldsymbol{I}_{12}^+\boldsymbol{P}_1^+,\ \boldsymbol{I}_{12}^+\boldsymbol{P}_1^-(=\boldsymbol{I}_{12}^+\boldsymbol{P}_2^-),\ \ -(\boldsymbol{I}_{12}^+\boldsymbol{P}_3^+\oplus\boldsymbol{I}_{12}^+\boldsymbol{P}_3^-). \tag{62}$$

If we had not made $\lambda_1 = 1$, all three expressions in (63) would be multiplied by λ_1. This is not significant for our purposes.

We have used the first parenthesis to make clear that the subscripts "2" of $\boldsymbol{P}$ also is represented. And the symbol $\oplus$ is used to indicate that, although $\boldsymbol{I}_{12}^+\boldsymbol{P}_3^+$ and $\boldsymbol{I}_{12}^+\boldsymbol{P}_3^-$ could be added to yield $\boldsymbol{I}_{12}^+$, we shall not do so. Thee reason is that, when dealing with solutions of exterior systems, they will be multiplied necessarily by different exponential factors, one with a positive exponent and the other with a negative one.

For the second one, we make $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 1$. It follows that $\lambda_5 = \lambda_6 = 0$, $\lambda_7 = \lambda_8 = -\lambda_3$. In this case we have

$$\boldsymbol{I}_{12}^-\boldsymbol{P}_1^+,\ \ \boldsymbol{I}_{12}^-\boldsymbol{P}_1^-(=\boldsymbol{I}_{12}^-\boldsymbol{P}_2^+),\ \ -(\boldsymbol{I}_{12}^-\boldsymbol{P}_3^+\oplus\boldsymbol{I}_{12}^-\boldsymbol{P}_3^-). \tag{63}$$

Once again parentheses have been used to illustrate the presence of all these subscripts of $\boldsymbol{P}$, which justifies are speaking of these parts.

To complete matters consider again Eq. (40). For $\mu = 0$, it reduces to

$$\sum_A \lambda_A(K+1)[d\boldsymbol{r}X_A] = \sum_A \pi_A. \tag{64}$$

$(K+1)$ acting on the non-scalars does not yield inhomogeneous terms (they are proper functions). And it yields zero when acting on the scalars. Hence

the left hand side of (65) is zero and so is, therefore, $\sum_A \pi_A$. Hence the co-value also is zero for those solutions with $\mu = 0$. They thus satisfy the equation

$$(K+1)d\boldsymbol{r}(\sum_A \lambda_A X_A) = 0. \tag{65}$$

We could have considered this equation as our starting point and solve for it.

We next bring (63) and (64) together and start to develop compact notation as follows.

Table 3. Constituent $\boldsymbol{I}_{12}\boldsymbol{P}$ idempotents

$a_1^3 = \boldsymbol{I}_{12}^+\boldsymbol{P}_1^+$	$a_2^3 = \boldsymbol{I}_{12}^+\boldsymbol{P}_1^-$	$a_3^3 = -\boldsymbol{I}_{12}'^+$
$b_1^3 = \boldsymbol{I}_{12}^-\boldsymbol{P}_2^+$	$b_2^3 = \boldsymbol{I}_{12}^-\boldsymbol{P}_2^-$	$b_3^3 = -\boldsymbol{I}_{12}'^-$

The superscripts of a and b are the missing index of I_{12}. We used primes on the right side of the equations for a_3^3 and b_3^3 to keep track of the remark made between Eqs. (63) and (64) about the $I_{12}'^{\pm}$ representing simplifications when the exponential factors multiplying the idempotents in solutions with symmetry of exterior systems are neglected.

One proceeds similarly for $\boldsymbol{I}_{31}$ and $\boldsymbol{I}_{23}$. We would thus prolong table 3 by placing table 4 above it.

Table 4. Constituent $\boldsymbol{I}_{22}^+\boldsymbol{P}$ and $\boldsymbol{I}_{31}^+\boldsymbol{P}$ idempotents

$a_1^1 = -\boldsymbol{I}_{23}'^+$	$a_2^1 = \boldsymbol{I}_{23}^+\boldsymbol{P}_2^+$	$a_3^1 = \boldsymbol{I}_{23}^+\boldsymbol{P}_2^-$
$b_1^1 = -\boldsymbol{I}_{23}'^-$	$b_2^1 = \boldsymbol{I}_{23}^-\boldsymbol{P}_3^+$	$b_3^1 = \boldsymbol{I}_{23}^-\boldsymbol{P}_3^-$
$a_1^2 = \boldsymbol{I}_{31}^+\boldsymbol{P}_3^-$	$a_2^2 = -\boldsymbol{I}_{31}'^+$	$a_3^2 = \boldsymbol{I}_{31}^+\boldsymbol{P}_3^+$
$b_1^2 = \boldsymbol{I}_{31}^-\boldsymbol{P}_1^-$	$b_2^2 = -\boldsymbol{I}_{31}'^-$	$b_3^2 = \boldsymbol{I}_{31}^-\boldsymbol{P}_1^+$

7 Inclusion of time translations

The operator associated with time translations is $-dt$ since the idempotent is

$$\boldsymbol{\epsilon}^{\pm} = \frac{1}{2}(1 \mp d\boldsymbol{t}) \tag{66}$$

and we have:

$$-d\boldsymbol{t}\boldsymbol{\epsilon}^{\pm} = -d\boldsymbol{t}(1 \mp d\boldsymbol{t}) = \pm\boldsymbol{\epsilon}^{\pm} \tag{67}$$

The total operator now is

$$T := (-d\boldsymbol{t})(K+1)d\boldsymbol{r}, \tag{68}$$

and its associated idempotents are of the type $\boldsymbol{\epsilon I P}$.

The idempotents given in tables 3 and 4 are now to be multiplied by $\boldsymbol{\epsilon}^+$ and $\boldsymbol{\epsilon}^-$. But, in the final result, we should make equal use of the two versions of the some idempotents that enter equations (32). The reason is that different versions of the same idempotents will be factors in different solutions of any given system of differential equations, where they would be multiplied by necessarily different exponentials (called phase factors in physics). In order to achieve that, we proceed as follows.

We reverse the signs in both superscripts of the idempotents $\boldsymbol{IP}$ in table 3. We thus obtain $\boldsymbol{I}_{12}^-\boldsymbol{P}_1^-, \boldsymbol{I}_{12}^-\boldsymbol{P}_1^+, \boldsymbol{I}_{12}^+\boldsymbol{P}_2^-$ and $\boldsymbol{I}_{12}^+\boldsymbol{P}_2^+$. Rewrite these idempotents in their alternative form: $\boldsymbol{I}_{12}^-\boldsymbol{P}_2^+$, $\boldsymbol{I}_{12}^-\boldsymbol{P}_2^-, \boldsymbol{I}_{12}^+\boldsymbol{P}_1^-$ and $\boldsymbol{I}_{12}^+\boldsymbol{P}_1^+$. These idempotents are the same ones as those that we started with, i.e. those of table 3 though in a different order. This suggests that we multiply the idempotents in table 3 by $\boldsymbol{\epsilon}^+$. At the same time, we first reverse the signs in the superscripts of the original idempotents and then multiply them by $\boldsymbol{\epsilon}^-$.

We rewrite them in a different way so that we can rewrite all of them easily from the top of our heads. Define

$$u_l^m = \boldsymbol{\epsilon}^+ a_l^m, \quad d_l^m = \boldsymbol{\epsilon}^+ b_l^m. \tag{69}$$

We further define $\overline{a}_l^m$ as a_l^m with reversion of the sign of each superscript. We first introduce symbols $\overline{u}_l^m$ and $\overline{d}_l^m$ as follows

$$\overline{u}_l^m = \boldsymbol{\epsilon}^- \overline{a}_l^m, \;\; \overline{d}_l^m = \boldsymbol{\epsilon}^- \overline{b}_l^m. \tag{70}$$

Corresponding to table 3, we would then have table 5.

Table 5. Constituent idempotents of type $\epsilon \boldsymbol{I}_{1,2}\boldsymbol{P}$

u/d	Subscript 1	Subscript 2	Subscript 3
u^3	$\boldsymbol{\epsilon}^+\boldsymbol{I}_{12}^+\boldsymbol{P}_1^+$	$\boldsymbol{\epsilon}^+\boldsymbol{I}_{12}^+\boldsymbol{P}_1^-$	$-\boldsymbol{\epsilon}^+\boldsymbol{I}_{12}'^+$
d^3	$\boldsymbol{\epsilon}^+\boldsymbol{I}_{12}^-\boldsymbol{P}_2^+$	$\boldsymbol{\epsilon}^+\boldsymbol{I}_{12}^-\boldsymbol{P}_2^-$	$-\boldsymbol{\epsilon}^+\boldsymbol{I}_{12}'^-$
$\overline{d}^3$	$\boldsymbol{\epsilon}^-\boldsymbol{I}_{12}^+\boldsymbol{P}_2^-$	$\boldsymbol{\epsilon}^+\boldsymbol{I}_{12}^+\boldsymbol{P}_2^+$	$-\boldsymbol{\epsilon}^-\boldsymbol{I}_{12}'^+$
$\overline{u}^3$	$\boldsymbol{\epsilon}^-\boldsymbol{I}_{12}^-\boldsymbol{P}_1^-$	$\boldsymbol{\epsilon}^-\boldsymbol{I}_{12}^-\boldsymbol{P}_1^+$	$-\boldsymbol{\epsilon}^-\boldsymbol{I}_{12}'^-$

All those idempotents are different. They are not so if we remove the factors $\boldsymbol{\epsilon}^{\pm}$. Notice that, once we remember the top line of this table, it is easy to reproduce the full table. We would proceed similarly with the generation of a table of constituent idempotents of types $\boldsymbol{\epsilon I}_{31}\boldsymbol{P}$, $\boldsymbol{\epsilon I}_{23}\boldsymbol{P}$, helping oneself with table 4 is needed.

8 Concluding Remarks

This paper is built on a few implicit premises. One of them is that there is much to be gained from looking at solutions with symmetry of exterior systems. Another one is that angular momentum should be preferred over individual components. Implicit is the premise that, when going beyond total angular momentum, as we have done, we should have $K+1$, rather than its components, multiply other operators.

The last two of those premises would have to be abandoned to connect with the physics as we know it. Operators $d\boldsymbol{t}/3$ and $d\boldsymbol{x}^{12}$ acting by product on the entries of table 5 yield familiar proper values and so does, therefore, their sum. But this involves returning to components. At least for spatial translations, we should keep them together. We have kept $d\boldsymbol{t}$ separately from $d\mathbf{r}$ because boosts does not play a role; it then appears that one should not treat $d\boldsymbol{t}$ and $d\mathbf{r}$ on an equal footing. It remains to be explored what a physics based on the contents of this paper would look like.

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